Extending partial automorphisms

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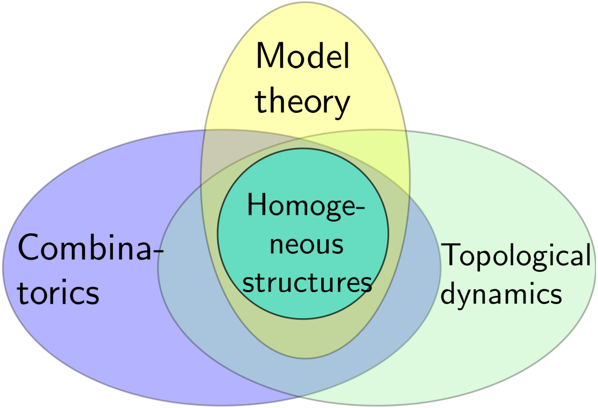
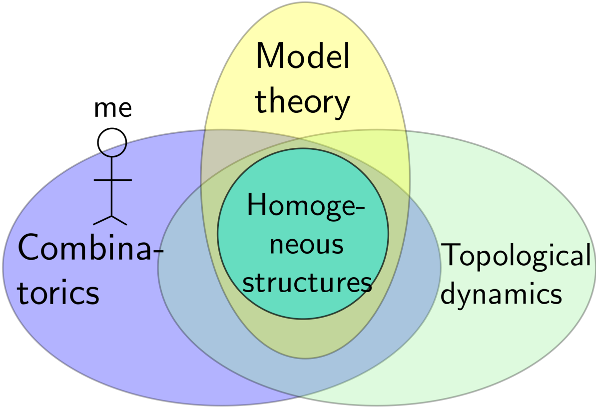
McGill Descriptive Dynamics and Combinatorics Seminar
24 Jan 2023


Model theory
Combinatorics
Homogeneous structures
Topological dynamics



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Intro for me

Let $\mathbf{G}$ be a graph. A partial function $f\colon V(\mathbf{G}) \to V(\mathbf{G})$ is a partial automorphism of $\mathbf{G}$ if f is an isomorphism of $\mathbf{G}[\mathrm{Dom}(f)]$ and $\mathbf{G}[\mathrm{Range}(f)]$ (subgraphs of $\mathbf{G}$ induced on $\mathrm{Dom}(f)$ and $\mathrm{Range}(f)$).

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- A graph **G** is vertex-transitive if every partial automorphism f with $|\mathrm{Dom}(f)| \leq 1$ extends to an automorphism of **G**.

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Example

- A graph **G** is vertex-transitive if every partial automorphism f with $|\mathrm{Dom}(f)| \leq 1$ extends to an automorphism of **G**.
- A graph **G** is edge-transitive (arc-transitive) if every partial automorphism f with $\mathrm{Dom}(f) = \{u, v\}$, where $uv \in E(\mathbf{G})$, extends to an automorphism of **G**.

Definition (EPPA, extension property for partial automorphisms)

Let **H** be a graph and let **G** be its **induced** subgraph. **H** is an EPPA-witness for **G** if every partial automorphism of **G** extends to an automorphism of **H**.

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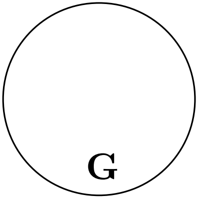

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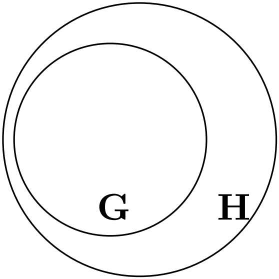

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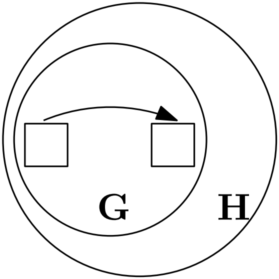

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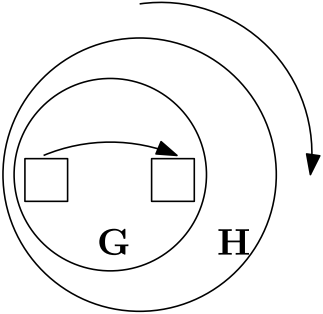

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Theorem (Hrushovski, 1992)

The class of all finite graphs has EPPA.

Intro for sophisticated people

Let $G \leq \mathrm{Sym}(\mathbb{N})$ be closed, i.e. $G = \mathrm{Aut}(\mathbf{M})$ for some structure $\mathbf{M}$.

Definition (HHLS'93, KR'07)

$\mathbf{M}$ has *n-generic automorphisms* if the action of G by diagonal conjugation has a comeagre orbit on G^n. It has *ample generics* if it has n-generic automorphisms for every $n \geq 1$.

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Theorem (HHLS'93, KR'07)

Let $\mathbf{M}$ be structure and $G = \mathrm{Aut}(\mathbf{M})$. Assume that $\mathbf{M}$ has APA and there is a sequence $G_0 \leq G_1 \leq \cdots \leq G$ of compact subgroups such that $G = \overline{\bigcup_i G_i}$. Then $\mathbf{M}$ has ample generics.

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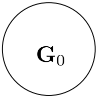

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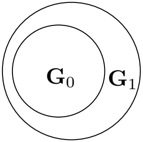

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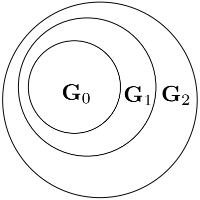

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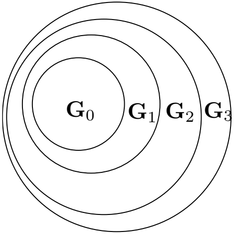

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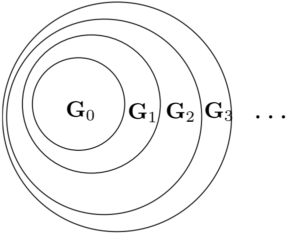

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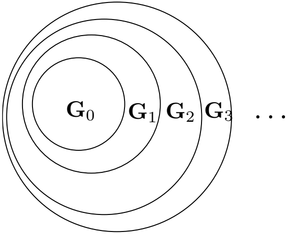

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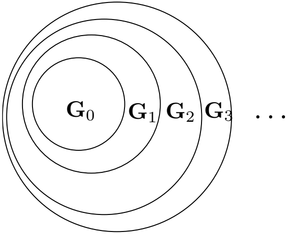

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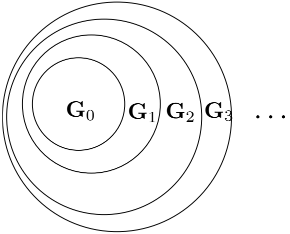

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Observation (Kechris, Rosendal'07)

The class of all substructures of a homogeneous structure $\mathbf{M}$ has EPPA if and only if $\mathrm{Aut}(\mathbf{M})$ can be written as the closure of a chain of compact subgroups.

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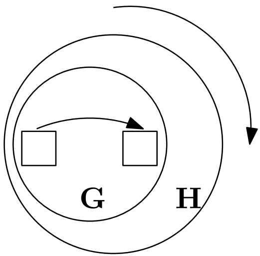

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There is **G** such that every EPPA-witness for **G** has at least $\Omega(2^{n/2})$ vertices, where $n = |V(\mathbf{G})|$.

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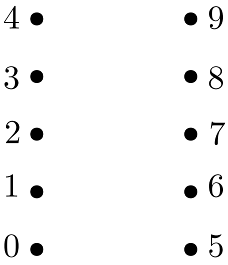

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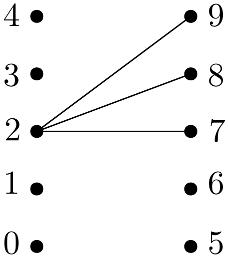

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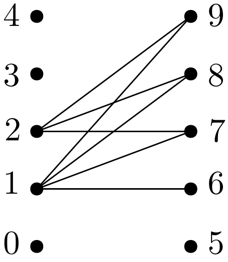

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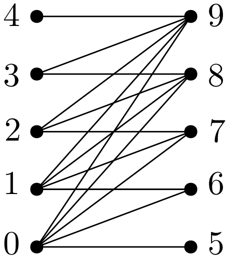

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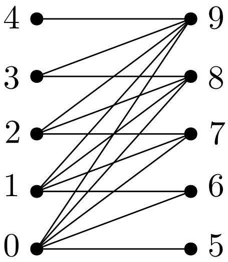

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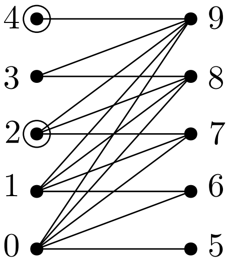

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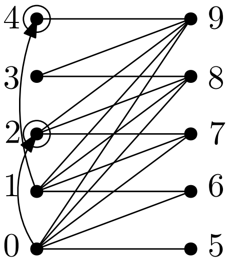

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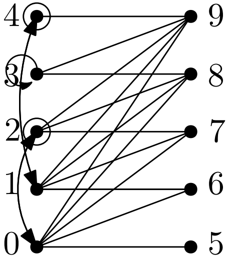

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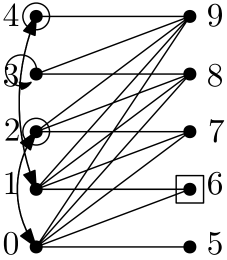

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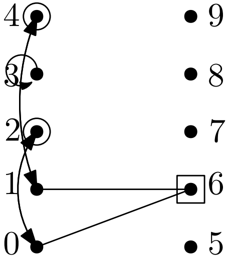

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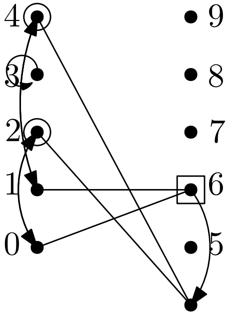

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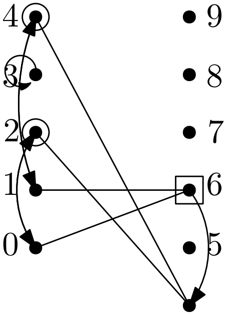
- Every permutation of $[m]$ is a partial automorphism of **G**.
- Pick any EPPA-witness and any $S \subseteq [m]$. There is a vertex v connected to S and not connected to $[m] \setminus S$.
- Hence every EPPA-witness for **G** has at least $\Omega(2^m)$ vertices.



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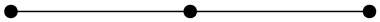

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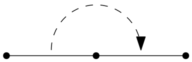

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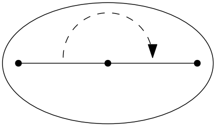

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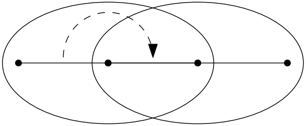

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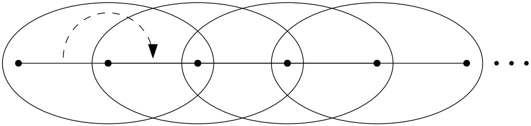

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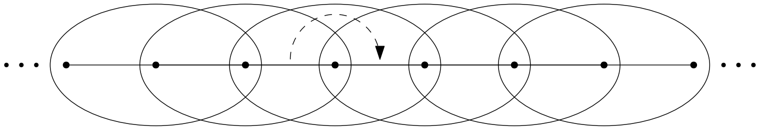

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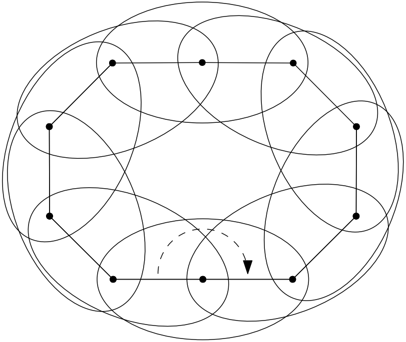
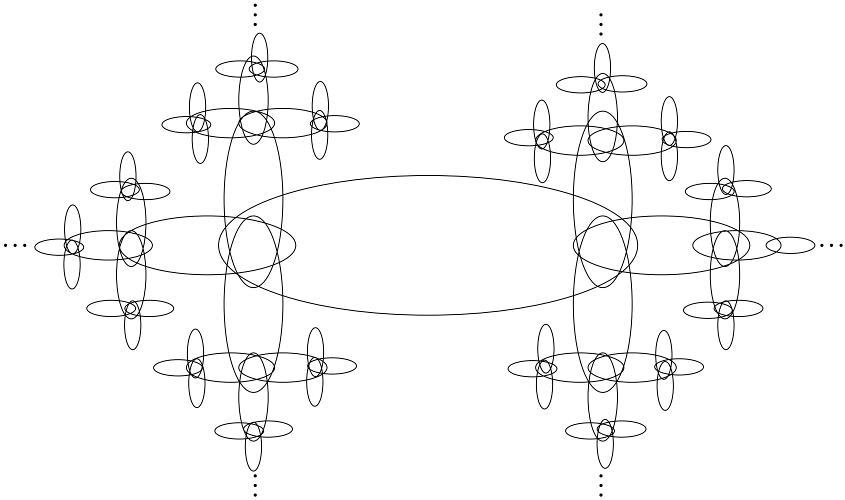


Multiple partial automorphisms are a different beast



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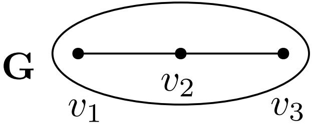

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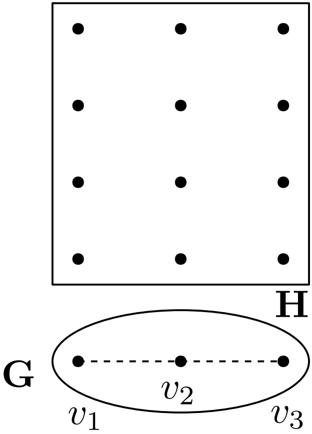

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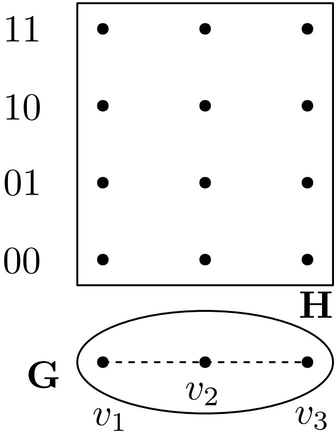

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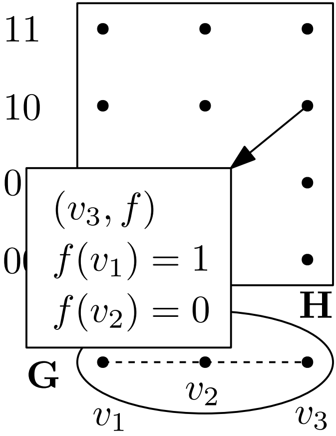

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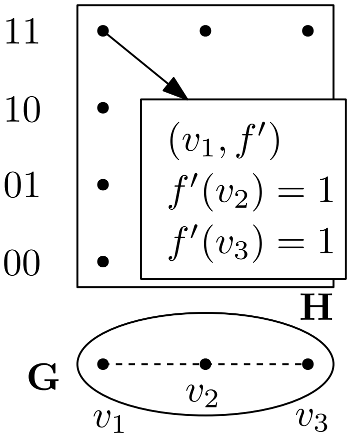

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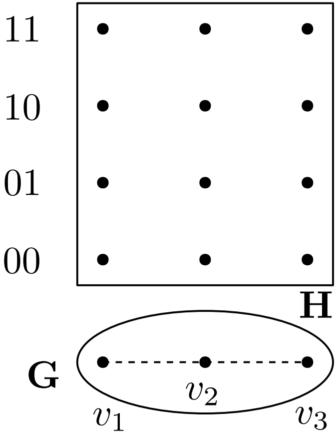

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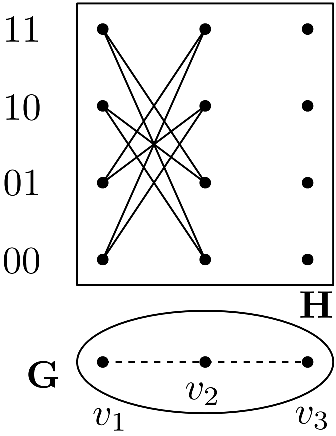

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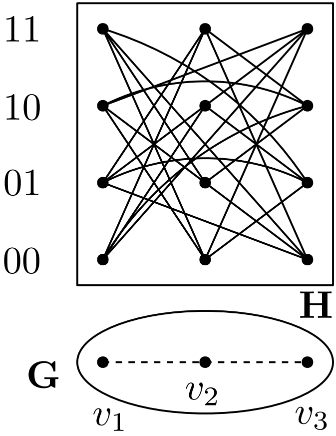

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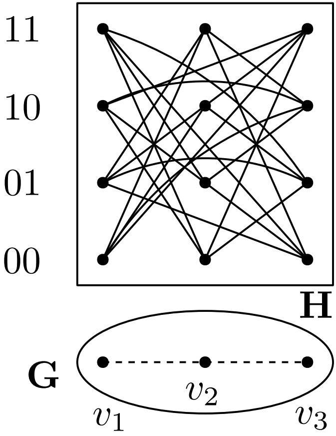

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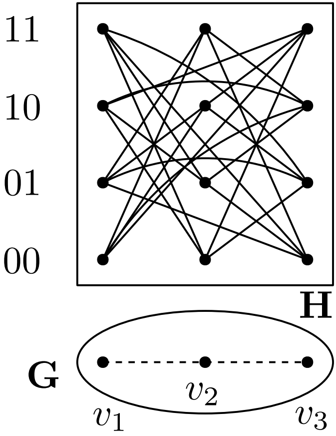

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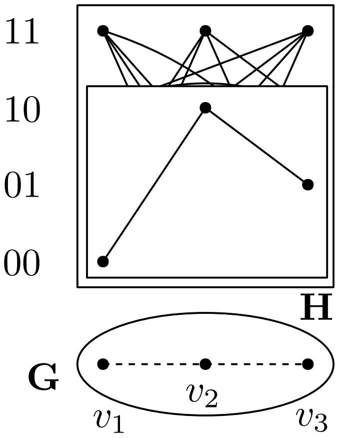

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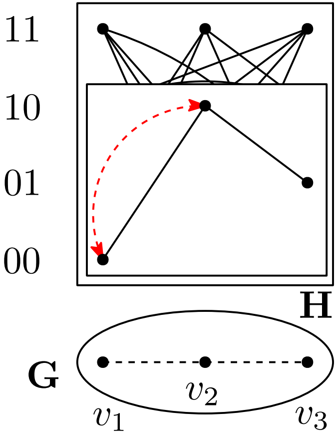

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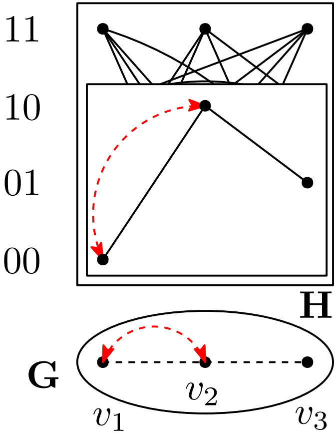

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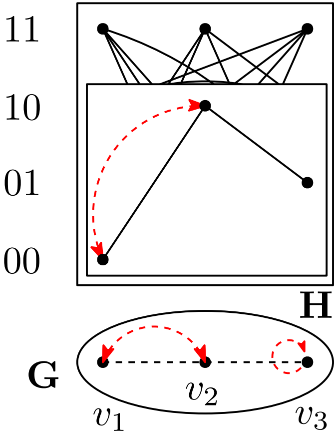

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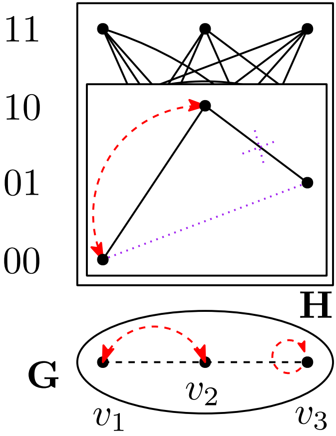

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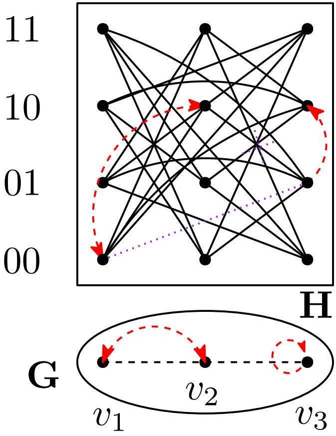

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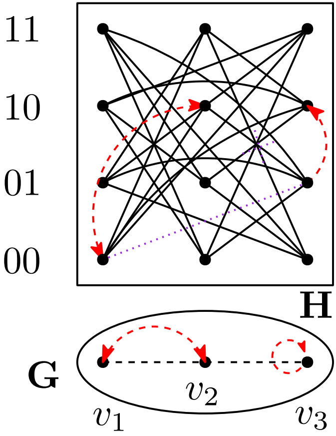

Remark

This can be straightforwardly generalised to arbitrary relational structures and less straightforwardly one can also add unary functions.

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- subgraphs of the finite homogeneous graphs [Gardiner76].

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- Graphs [Hrushovski, 1992], K_n-free graphs [Hodkinson–Otto, 2003]
- Relational structures (with forbidden cliques) [Herwig, 2000], [Hodkinson–Otto, 2003]
- Metric spaces [Solecki, 2005; Vershik, 2008], also [Conant, 2019]
- Two-graphs [Evans–Hubička–K–Nešetřil, 2018]
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ALL THOSE EPPA CLASSES (STRENGTHENINGS OF THE HERWIG–LASCAR THEOREM)

JAN HUBIČKA, MATĚJ KONEČNÝ, AND JAROSLAV NEŠETŘIL

ABSTRACT. Let **A** be a finite structure. We say that a finite structure **B** is an extension property for partial automorphisms (EPPA)-witness for **A** if it contains **A** as a substructure and every isomorphism of substructures of **A**

- Strengthens [Herwig–Lascar, 2000], [Hodkinson–Otto, 2003] and [Evans–Hubička–Nešetřil, 2021].
- General constructions of EPPA-witnesses, sufficient conditions for a class to have EPPA.
- *Most of the known relational homogeneous structures.*
- Conditions almost isomorphic to [Hubička–Nešetřil, 2017] for Ramsey properties.

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Does the class of all finite tournaments have EPPA?

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5. Extend it to a permutation of E respecting the partial automorphism. $\square$

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If the maximum degree of $\mathbf{G}$ *is* Δ*, then there is an EPPA-witness on* $\mathcal{O}\left((\Delta n)^{\Delta}\right)$ *vertices.*

Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is k-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{k}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
- Every permutation of E induces an automorphism of $\mathbf{H}$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism. □

For non-k-regular graphs, add "half-edges" to make them regular.